PROBLEMS IN LOW DIMENSIONAL

MANIFOLD THEORY

Edited by Rob Kirby

This is a preliminary list, mostly obtained from the two problem sessions at the Stanford conference (August 1976). I hope that all of you will

1) check carefully all problems attributed to you,
2) read all the problems so that you can be reminded of cogent remarks you want to make or further problems you want to send me,
3) send me material in well written form, so that I can use it verbatim if necessary,
4) show your copy to others who may not have received one.

Having gone this far with the problem list, I am inclined to make it fairly complete. I've followed no rule with regard to famous problems, e.g., the Poincaré conjecture is not listed, the Smith conjecture is. Since students will be amongst the biggest users, perhaps no problem is too well known. Here's a criterion for listing a problem: the author should have worked on it and should feel sure at least n ($n > 0$) others would, too.

I hope to write the final list in November, so make that a deadline for contributions.

§1 KNOT THEORY

Two earlier lists of problems are given by Fox in Topology of 3-Manifolds, Prentice-Hall, 1962, and Neuwirth in his book, Knot Groups, Princeton University Press, 1965.

PROBLEM 1: (R. H. Fox and J. Birman) Let G be the group of a non-trivial knot, and let x be a meridian in G. Let $N(x^2)$ be the normal closure of x^2 in G. Conjecture: $G/N(x^2)$ is never abelian.

PROBLEM 2: (L. Moser) Is there a geometrical characterization of knots whose groups have one relator?

Remarks: The groups of 2-bridge knots are presented on 2 generators and one relator where the generators are meridians. The groups of torus knots are also presented on 2 generators with one relator but the generators are not meridians.

PROBLEM 3: (Cappell and Shaneson) Is every knot, whose group is generated by 2 meridians, actually a 2-bridge knot? Same for n meridians and n-bridge knots.

PROBLEM 4: (T. Matumoto) Suppose the band connected sum of a trivial link (of two components) is the trivial knot. Is the band isotopic to the trivial band?

PROBLEM 5: Does every non-trivial knot K have Property P; that is, does Dehn surgery on K always give a non-simply connected manifold?

Remarks: Knots with Property P include. . .(someone please send a list). It is also interesting to know which knots never give S^3.

PROBLEM 6: (Kirby) Is the homology 3-sphere obtained by $\pm$ 1-surgery on a knot always prime? Is every prime homology 3-sphere obtained that way?

Remark: Who knows a non-prime example which is not obtained by $\pm$ 1-surgery on a knot?

PROBLEM 7: Does every non-trivial knot K have Property R, that is, does surgery on K (with 0 framing) not give $S^1 \times S^2$?

Remarks: If 0-surgery on K gives $S^1 \times S^2$, then K must be slice (R. Kirby-P. Melvin), and. . . (list please).

PROBLEM 8: (R. Edwards) If ℓ is the longitude of K in S^3-K, is it possible that (a) ℓ is algebraically a product of commutators of itself, i.e., does there exist $a_1,\ldots,a_n \in \pi_1(S^3-K)$ such that $\ell = [a_1,\ell][a_2,\ell]\ldots[a_n,\ell]$;

(b) ℓ is geometrically such a product?

Remark: F. Laudenbach has shown that (b) is not possible for $n=1$. D. Galewski has proved that (a) implies that 0-surgery on K gives a homotopy $S^1 \times S^2$ (in Necessary and Sufficient Conditions to Obtain $S^2 \times S^1$ by Surgery on a Knot, preprint, Georgia).

PROBLEM 9: (S. Akbulut and R. Kirby) Conjecture: If 0-framed surgery on two knots gives the same 3-manifold, then the knots are concordant.

Remarks: This is true if one knot is the unknot (Kirby-Melvin) (see PROBLEM 7). If homotopy 4-spheres are spheres, then it is true if one knot is slice. In general all known concordance invariants of the two knots are the same.

PROBLEM 10: (D. Goldsmith) Do there exist distinct knots K and K' in S^3, all of whose cyclic branched covers are homeomorphic?

PROBLEM 11: (D. Goldsmith) Let $M^3 \xrightarrow{\pi} S^3$ be an n-fold cyclic branched cover of S^3 along a knot K. Let A be an unknot in $S^3 - K$. If K is a closed braid about A, then $\pi^{-1}(A)$ is a fibered knot in M^3. Is the converse true?

PROBLEM 12: (Cappell and Shaneson) Is every closed, oriented

3-manifold the dihedral covering space of a ribbon knot?

PROBLEM 13: (Cappel and Shaneson) Let M_α be the p-fold dihedral cover of a knot α. Let $\alpha_0, \alpha_1, \ldots, \alpha_r$, $r = (p-1)/2$, be the branching curves in M_α where α_0 has branching index 1. Let $v_{i0} = \ell(\alpha_i, \alpha_0)$. Prove that $v_{i0} \equiv 2\ (4)$ if M_α is a Z/2-homology sphere.

Remarks: This is known for $p = 4k+3$, but not $p = 4k+1$. It is known that $\sum_{i=1}^{r} v_{i0} \equiv p-1\ (4)$. For 2-bridge knots, $v_{i0} = 2$. (References, please.)

PROBLEM 14: (Cappell and Shaneson) The μ-invariant formula of Cappel-Shaneson (BAMS, 81 (1975), 559-561) detects non-ribbon knots. Does it detect non-slice knots as well? Relate this to the Casson-Gordon invariant.

PROBLEM 15: (Cappell and Shaneson) Are the classical PL and TOP knot concordance groups the same?

Remarks: Clearly $C^{PL} \to C^{TOP}$ is onto. This question may be easier than the hauptvermutung for $B^2 \times R^2$.

PROBLEM 16: (Y. Matsumoto) Let $\mathcal{H}_A$ = {knots in a homology-S^3 which bounds a PL acyclic 4-manifold, modulo concordance}. Is the natural map $C^{PL} \to \mathcal{H}_A$ an isomorphism?

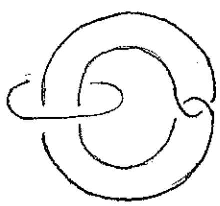
PROBLEM 17: (C. McA. Gordon) Does the classical knot concordance group contain any non-trivial elements of finite order other than 2 ?

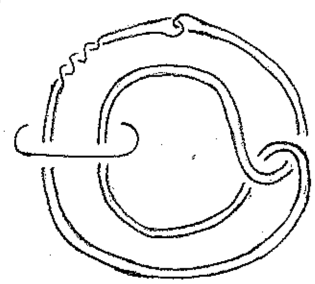
PROBLEM 18: (L. Taylor) If a knot has Alexander polynomial equal to one, is it a slice knot?

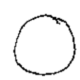
Remarks: Any such knot is algebraically slice. These knots have no dihedral covers so the Casson-Gordon methods will not prove they are not slice.

PROBLEM 19: Conjecture: The double of a knot is slice $\Leftrightarrow$ the knot is slice.

PROBLEM 20: (A. Casson) Drawn below is the Whitehead link and a double of the Whitehead link.





This construction can be iterated by replacing  by  or  ; call the n^{th} iterate W_n . Is any W_n null-concordant?

Remarks: If not, there exists an end of a 4-manifold, $Q \overset{P}{\cong} S^2\times S^2$-pt , which is fake (A. Casson). It is also interesting to know if W_n is null-concordant in some contractible 4-manifold.

PROBLEM 21: (M. Scharlemann) Are there knots $f: S^1 \to S^3$ such that for any locally flat concordance $F: S^1 \times I \to S^3 \times I$ the map $\pi_1(S^3 - f(S^1)) \to \pi_1(S^3 \times I - F(S^1 \times I))$ is injective?

Conjecture: This is true for torus knots.

Remark: This is true for torus knots if F must be a fibered concordance.

PROBLEM 22: (L. Kauffman) Does link concordance imply link homotopy?

PROBLEM 23: (S. Akbulut) Given a knot K, an "algebraically one strand" is a way of embedding $S^1 \times B^2$ (unknotted) in S^3 with $K \subset S^1 \times B^2$, $K \cap \text{point} \times B^2$ algebraically equal to one, and K not isotopic in $S^1 \times B^2$ to a knot K' with $K' \cap \text{point} \times B^2 =$ one point. Conjecture: There exists a knot K and an algebraically one strand such that no matter what knot is tied in the strand (in $S^1 \times B^2$), the new K is not slice in a homotopy 4-ball.

If the conjecture is true, then there exists a knot in the boundary of a contractible 4-manifold which does not bound a PL 2-ball.

PROBLEM 24: (Akbulut-Kirby) Define a (k,ℓ) twist on a knot

K in S^3 as follows: imbed a 2-disk in S^3 transverse to K and intersecting K algebraically ℓ times; then give K k full twists around the normal to the 2-disk. Given an Arf invariant zero knot K, is there some $(\pm 1, \pm 1)$ twist changing K into a slice knot? The unknot? (Surely the answer is no.)

Remarks: It is possible to change K into an algebraically slice (Seifert matrix concordant to zero) knot. The question is motivated by the statement: if every homotopy CP^2 is diffeomorphic to CP^2, then there is a $(\pm 1, \pm 1)$ twist changing K to a slice knot iff ± 1 surgery on K gives a homology sphere bounding a contractible 4-manifold.

PROBLEM 25: (J. Levine) A general question is what groups π are fundamental groups of the complement of some knotted S^2 in S^4? Recall that a group π has weight 1 if it is normally generated by one element, and deficiency one if it has a presentation $\{x_1, \ldots, x_n, t: R_1, \ldots, R_n\}$ with one more generator than relation.

(1) Given π such that $H_1(\pi) = Z$, π has weight one and deficiency one, then π is the group of an S^2 in a homotopy 4-sphere (Kervaire (ref?)). Which of these are realizable by knots in S^4? π is realizable if the induced presentation of the trivial group defined by setting $t=1$ is trivializable by Andrews-Curtis moves (ref?).

(2) Let $\Lambda = Z[t, t^{-1}]$ and let the Λ-module A of an

$S^2 \to S^4$ be π'/π'' with the induced action of $\pi/\pi' = z$. Which Λ-modules are realizable? If A is Z-torsion free (implied by deficiency one), the answer is known since there are enough deficiency one π to get all such A's.

PROBLEM 26: (S. Lomonaco) Does there exist a locally flat 2-sphere in 4-space such that the deficiency of the fundamental group of its complement is neither 0 nor 1 ?

PROBLEM 27: (Gordon) Can a branched cyclic cover of a (locally flat) knot $S^n \to S^{n+2}$ ever be a $K(\pi,1)$, for $n \geqslant 2$?

§2 3-MANIFOLDS

PROBLEM 28: (H. Hilden and J. Montesinos) Is every homology 3-sphere the double branched covering of a knot in S^3 ?

Remarks: It is known that $S^1 \times S^1 \times S^1$ is not a double branched covering (Fox) and that $S^1 \times T_g$ (T_g = surface of genus g) is not a cyclic branched covering (Montesinos), but the arguments depend on a nontrivial first homology group.

PROBLEM 29: (J. Birman) Let K be a knot in S^3 and $M(K)$ its 2-fold branched covering space. To what extent do topological properties of M determine K ? More generally, describe the equivalence class $[K]$ of K under the relation $K_1 \cong K_2$ if $M(K_1)$ is homeomorphic to $M(K_2)$.

Remarks: (1) If K is a 2-bridge knot, then $M(K)$ determines K .

(2) If $M(K)$ is composite, then K is composite (Kim and Tollefson, "Splitting PL involutions on 3-manifolds").

(3) The bridge index of $K \leq$ Heegard genus of M (J. Birman and H. Hilden, Heegard splittings of branched coverings of S^3 , TAMS, (1975)).

(4) There are examples of distinct prime 3-bridge knots which have homeomorphic 2-fold covering spaces (J. Birman, Gonzalez-Acuña and J. Montesinos, Heegard splittings of prime 3-manifolds are not unique, to appear in Mich. Math. J.).

PROBLEM 30: (F. Waldhausen) Given a closed, orientable M^3, define the space of Heegard splittings $\mathcal{H}(M)$ as the simplicial category which in degree k is $\mathcal{H}(M)^k$ where the objects in $\mathcal{H}(M)^k$ are k-parameter families of Heegard splittings of M and the morphisms in $\mathcal{H}(M)^k$ are k-parameter families of standard handle additions. It is known (Reidemeister-Singer theorem) that $\pi_0(\mathcal{H}(M))$ = point.

$\mathcal{H}(M)$ can be filtered by requiring that the Heegard splitting has genus $\leqslant k$; thus $\mathcal{H}(M) = \bigcup_k \mathcal{H}_k(M)$. It is known that $\pi_0(\mathcal{H}_k(S^3))$ = point for all k (F. Waldhausen). What else can one say?

PROBLEM 31: (S. Smale) Conjecture: $\mathrm{Diff}^+(S^3)$ is homotopy equivalent to $SO(4)$.

Remark: $\pi_0(\mathrm{Diff}^+(S^3)/SO(4)) = 0$ (Cerf, Springer Lecture Notes 53 (1968)).

PROBLEM 32: (A. Hatcher) Compute π_0 $\mathrm{Diff}(L^3)$, the space of diffeomorphisms of a lens space.

PROBLEM 33: (Hilden and Montesinos) Every closed, orientable 3-manifold can be constructed as follows: let F_1 and F_2 be disjoint, closed surfaces (not necessarily orientable) in S^3 . Take three copies of $(S^3;F_1,F_2)$, called $S_a^3,S_b^3,S_c^3,F_{1a},\ldots$ etc. Split S_a^3 along F_1 , S_b^3 along F_1 and F_2 , and S_c^3 along F_2 . Then glue one side of F_1 in S_a^3 to the other side in S_b^3 , and one side of F_2 in S_c^3 to the other side in S_b^3 . Question: can the surfaces be chosen to be orientable?

<u>PROBLEM 34</u>: Classify imbeddings of surfaces in S^3 .

<u>Remarks</u>: For S^2 , the Schoenflies theorem classifies. For T^2 , any imbedding bounds an $S^1 \times B^2$ (Alexander, P.N.A.S. 10 (1924), 6), so the classification "reduces" to knot theory.

<u>PROBLEM 35</u>: (D. Rolfsen) Theorem: Every closed orientable M^3 contains a fibered knot K ; i.e., there exists a fibration $f: M-K \to S^1$ and f is standard on a deleted tubular neighborhood of K (Gonzalez-Acuña, Myers (ref?)). This is Winkelkemper's open book decomposition but with connected binding. Note that K is homologically trivial in M . Question: what elements of $\pi_1(M)$ are represented by fibered knots? Links?

<u>PROBLEM 36</u>: (A. Hatcher) Is an irreducible h-cobordism from P^2 to itself a product?

<u>Remark</u>: This is known if its double cover is $S^2 \times I$ (ref?).

<u>PROBLEM 37</u>: Let M^3 be a closed $K(\pi,1)$ with π infinite. Does M , or a homotopy equivalent closed 3-manifold, have R^3 as universal cover?

<u>PROBLEM 38</u>: Let M^3 be a closed $K(\pi,1)$. Find examples of such 3-manifolds which are neither Seifert fiberings nor

sufficiently large. Does M have a finite cover which is sufficiently large?

PROBLEM 39: (Jaco) A sufficiently large 3-manifold is atoroidal if it contains no essential annuli or tori. What groups appear as π_1 of an atoroidal manifold?

Remark: Such manifolds are determined by their fundamental groups (Johannson).

PROBLEM 40: (Jaco) Are 3-manifold groups Hopfian (any epimorphism $G \to G$ is monic)? Residually finite (given $g \in G$, $g \neq 1$, there is a representation λ of G to a finite group for which $\lambda(g) \neq 1$)?

PROBLEM 41: (P. A. Smith) Conjecture: If $h: S^3 \to S^3$ is a period p homeomorphism with fixed point set S^1, then S^1 is unknotted.

Remark: True for p even (F. Waldhausen, Topology 8 (1969), 81-91).

PROBLEM 42: (C. Thomas) Let Z/r act freely on S^3. Compute the Reidemeister torsion of the quotient space. In particular, is every such quotient simple homotopy equivalent to a lens space?

Remark: At the Poincare complex level one can geometrically realize non-linear torsions by varying the attaching map for e^3 .

PROBLEM 43: (C. Thomas) In the case of the binary dihedral group Q_8 , any quotient is simple homotopy equivalent to the (unique) linear quotient, since $Wh(Q_8) = 0$ (Keating). Is the homeomorphism to $L(4,\pm 1)$, which exists on a double cover, the lift of a homeomorphism on the base?

§3 4-MANIFOLDS

PROBLEM 44: (Kirby) What integral, unimodular, symmetric bilinear forms are the intersection forms of simply connected closed 4-manifolds?

Odd, indefinite forms are represented by connected sums of CP^2 and $\overline{CP}^2$, but little is known otherwise. In particular, is $E_8 + \langle 1\rangle$ (the odd definite form of index 9), or $E_8 + E_8 + n\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $n \leqslant 2$, or Γ_{16} (the other index 16 even definite form) represented by a manifold?

The homology sphere $\Sigma(2,7,13)$, obtained either as the link of the complex singularity $x^2+y^7+z^{13} = 0$ or by surgery with $+1$ framing on the (2,7)-torus knot, bounds a manifold with form Γ_{16}; does it bound an even form of rank $\leqslant 4$?

It is known that $\Sigma(2,7,13)$ lies in the Kummer surface with Γ_{16} on one side, and an even form of rank 6 on the other.

PROBLEM 45: What homology 3-spheres with Rohlin (μ-) invariant zero bound contractible 4-manifolds? Acyclic 4-manifolds?

Remarks: Akbulut's candidate for one which doesn't bound an acyclic manifold is $\Sigma(2,3,11)$ which can be gotten by $+1$ surgery on the knot  .

For the purpose of triangulating non-PL manifolds one needs an example of a Rohlin invariant 1 homology 3-sphere whose connected sum with itself bounds an acyclic manifold, and an example of two Rohlin invariant 1 homology 3-spheres whose connected sum bounds a contractible manifold (Galewski-Stern, T. Matsumoto, R. Edwards).

PROBLEM 46: Conjecture: Two simply-connected, closed 4-manifolds are homeomorphic if they are homotopy equivalent.

Remark: They are h-cobordant (Wall, J. London M. S. 39 (1964), 141-149).

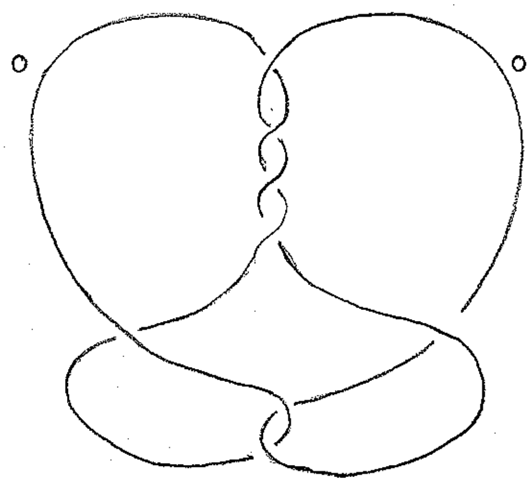
PROBLEM 47: (Y. Matsumoto) Is Scharlemann's "fake" $S^1 \times S^3 \# S^2 \times S^2$ an exotic manifold or an exotic self-homotopy equivalence of $S^1 \times S^3 \# S^2 \times S^2$? (M. Scharlemann, Duke M. J. 43 (1976), 33-40.)

PROBLEM 48: (Cappell and Shaneson) There are homotopy RP^4's which are not diffeomorphic to RP^4 (Cappel and Shaneson, Ann. Math. 104 (1976),). Which of these homotopy RP^4's are homeomorphic (diffeomorphic) and which are homeomorphic to RP^4 ?

Remark: Some of the homotopy RP^4's have double covers which are diffeomorphic to S^4 (Akbulut and Kirby).

PROBLEM 49: (S. Akbulut and R. Kirby) Does every diffeomorphism of the boundary of a contractible 4-manifold X^4 extend over X^4 ?

Remarks: If not, there is a counterexample to the relative h-cobordism theorem in dimension 5 . Here is a candidate for a diffeomorphism which does not extend: in the symmetric link below, we can add 2-handles (to B^4) to both circles with framing 0 . The boundary of this 4-manifold has an obvious involution obtained by switching circles. Let the contractible manifold X^4 be obtained by surgering one of the two obvious 2-spheres; X^4 is a well known Mazur manifold.



PROBLEM 50: (Kirby) Does every simply-connected, closed 4-manifold have a handle body decomposition without 1-handles? Without 1 and 3-handles?

Remark: Because there are non-trivial groups G which cannot be trivialized by adding the same number of generators and relations (Gerstenhaber and Rothaus, P.N.A.S. 48 (1962), 1531-33), there are contractible 4-manifolds V^4 , with $\pi_1(\partial V^4) = G$, that require 1-handles (Casson). On the other hand, a non-singular, algebraic surface in CP^3 needs no 1-handles (L. Rudolph, Topology 14 (1976), 301-303).

PROBLEM 51: (S. Weintraub) Does every simply-connected, closed 4-manifold have a basis for H_2 consisting of PL imbedded 2-spheres? Smooth?

Remark: Yes, in the PL case, if there is a 2-dimensional spine, or if there are no 1-handles.

PROBLEM 52: (Gordon) Let Σ^3 be a homology 3-sphere which bounds an acyclic 4-manifold V^4 such that $\pi_1(\Sigma^3) \to \pi_1(V^4)$ is surjective. Let K be a knot in Σ^3. Define K to be homotopically ribbon in V if there is a smoothly, imbedded B^2 in V, $\partial B^2 = K$, such that $\pi_1(\Sigma^3 - K) \to \pi_1(V - B^2)$ is surjective.

(a) Does " K slice in V " imply " K homotopically ribbon in V "?

(b) Does (a) hold for at least contractible V ?

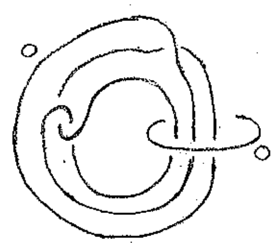
Remarks: A yes answer to (b) implies the existence of knots $K \subset \partial W$, W contractible, such that K does not bound an imbedded PL disk in W.

The classical "slice implies ribbon" conjecture splits into two parts, "slice implies homotopically ribbon (in B^4)" and "homotopically ribbon implies ribbon".

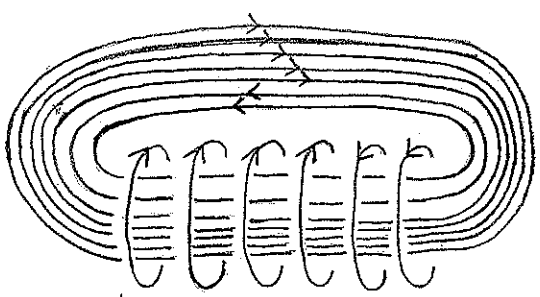
PROBLEM 53: (Y. Matsumoto) Does there exist a smooth, compact W^4, homotopy equivalent to S^2, which is spineless, i.e., contains no PL imbedded S^2 representing the generator of $H_2(W)$?

Remark: There is such an example for T^2 instead of S^2 (Y. Matsumoto, ref?).

PROBLEM 54: (Y. Matsumoto) Let M^4 be obtained by attaching 2-handles to B^4 along the Mazur link with 0 framings. Does there exist a smooth imbedding $S^2 \times S^2 \xrightarrow{\cong} M^4$?



PROBLEM 55: (Y. Matsumoto) Does the following link in S^3 bound a smooth punctured sphere in B^4? If so, $(2,3) \in H_2(S^2 \vee S^2;Z)$ is represented by a smooth S^2. Can it be represented by a torus? Is there a homology $S^2 \times S^2$ in which $(2,3)$ is represented by a smooth S^2?



PROBLEM 56: (L. Taylor) Construct a fake Hopf bundle by realizing $\Gamma = 3\gamma_0 + \gamma_1 + \ldots + \gamma_8$ in $CP^2 \# 8\overline{CP}^2$ by a PL imbedded

sphere and taking a regular neighborhood (γ_i is the generator of H_2 of CP^2 or $\overline{CP}^2$; the "Hopf bundle" is $B^4 \cup$ 2-handle attached to the trefoil knot with $+1$ framing). Twice the core of this "Hopf bundle" can be represented by a smoothly imbedded double torus. Can it be represented by a torus? A sphere?

Remarks: $r\Gamma$ cannot be represented by a smoothly imbedded sphere if $r=1$ (Kervaire-Milnor, P.N.A.S. 47 (1961), 1651-1657) or if $r \geqslant 3$ (Tristam, Proc. Cam. Phil. Soc. 66 (1969), 251-264; W.-C. Hsiang and R. H. Szczarba, Proc. Sym. Pure Math., AMS 22 (1970), 97-103). The double branched cover of this "Hopf bundle" along the imbedded surface can be used to construct a spin manifold of index 16 and betti number 22 (double torus), 20 (torus), 18 (sphere).

PROBLEM 57: (Kirby) Let $f: S^2 \to CP^2$ be a smooth imbedding which represents the generator of $H_2(CP^2;Z)$. Conjecture: $(CP^2, f(S^2))$ is pairwise diffeomorphic to (CP^2, CP^1) . Perhaps $f(S^2)$ is even isotopic to CP^1 .

Remark: The conjecture may be easier than the Poincare conjecture which implies it.

PROBLEM 58: Does there exist a closed, almost parallelizable TOP 4-manifold of index 8 ?

PROBLEM 59: Does there exist a manifold proper homotopy equivalent (or even homeomorphic) to $S^3 \times R$, but not diffeomorphic?

Casson has shown that either such a manifold exists or another manifold, $Q^4 \overset{P}{\cong} S^2 \times S^2 - pt$, exists having a fake end (see PROBLEM 20).

PROBLEM 60: (Schoenflies) If S^3 is PL imbedded in S^4, then its closed complements are PL 4-balls.

Remark: Note that they are TOP 4-balls since the S^3 is (PL) locally flat.

PROBLEM 61: (M. Cohen) Does there exist a 4-dimensional h-cobordism $(W^4; M_1^3, M_2^3)$, with any π_1, such that W^4 is not $M_1 \times I$?

PROBLEM 62: (A. Hatcher) On the torus T^n, $n \geqslant 5$, there are many homeomorphisms concordant but not isotopic to the identity; (A. Hatcher, these proceedings). Are there such examples on T^4 ?

PROBLEM 63: (T. Matsumoto) Let S be a simply connected complex surface with $c_1 = 0$ (2). Then there exists a complex line bundle $\frac{1}{2}K$ such that index $S = -8(2 \dim H^0(S; \mathcal{O}(\frac{1}{2}K)) - \dim H^1(S, \mathcal{O}(\frac{1}{2}K))$. Is index $S \leqslant 0$? Is $H^1(S, \mathcal{O}(\frac{1}{2}K)) = 0$?

§4 MISCELLANY

PROBLEM 64: (M. Cohen) Let $\mathcal{P}$ and $\mathcal{P}'$ be finite presentations of a given group π. Let K and $K_{\prime}$ be the 2-dim CW-complexes associated to these presentations. Consider the assertions:

A) $K_{\mathcal{P}} \cong K_{\mathcal{P}'}$ (homotopy equivalence)

B) $K_{\mathcal{P}} \wedge K_{\mathcal{P}'}$ (simple homotopy equivalence)

C) $K_{\mathcal{P}} \overset{3}{\wedge} K_{\mathcal{P}'}$ (simple homotopy equivalence by moves of dimension $\leqslant 3$)

D) $\mathcal{P}$ can be changed to $\mathcal{P}'$ by Andrews-Curtis moves (i.e., we can change the presentation $\mathcal{P} = \{x_1, \ldots, x_n : R_1 \ldots R_n\}$ in these ways: (i) $R_i \to R_i^{-1}$,

(ii) $R_i \to R_i R_j$,

(iii) $R_i \to wR_i w^{-1}$, w any word,

(iv) add generator x_{n+1} and relation wx_{n+1};

note relations cannot be remembered). Then $D \Rightarrow C \Rightarrow B \Rightarrow A$ and $C \Rightarrow D$ (??). D fails for the trefoil group (Dunwoody, Bull. Lon. Math. Soc. 4 (1972), 151-55). What other implications hold?

PROBLEM 65: (Lickorish) Let K be a contractible 2-complex.

A) Conjecture (Zeeman): $K \times I$ collapses to a point.

B) Conjecture: K 3-deforms to a point, i.e., there exists a 3-complex L such that $K \nearrow L \searrow pt$.

C) Conjecture: The unique 5-dim regular neighborhood of K^2 in R^5 is B^5.

D_0) Conjecture: Any presentation of the trivial group can be changed to the trivial presentation by Andrews-Curtis moves.

Remarks: Conjecture A implies the Poincare Conjecture. Conjecture C is equivalent to knowing whether the boundary is S^4. $A \Rightarrow B \Rightarrow C$ and $B \Leftrightarrow D$. Conjecture D is false for non-trivial groups (see D) of PROBLEM 64). Possible counterexamples are $\{a,b: a^{-1}b^2a=b^3, b^{-1}a^2b=a^3\}$ and $\{a,b,c: [a,b]b=[b,c]c=[c,a]a=1\}$. It is not known whether the regular neighborhoods in R^5 of the corresponding 2-complexes are B^5.

PROBLEM 66: (M. Freedman) Let A and B be torsion free groups with $H_1(A*B) = 0$ or Z. Conjecture: $A*B$ is not in the normal closure of a single element.

Remarks: Knot complements are irreducible and π_1 is normally generated by a meridian. The conjecture would give an algebraic proof of irreducibility, and would imply that $\pm$ 1-surgery on a knot gives an irreducible manifold (see PROBLEM 6).

$Z*G$ is not normally generated by a single element if G represents non-trivially into a compact Lie group (Gerstenhaber and Rothaus, P.N.A.S. 48 (1962), 1531-33). Note that if $G = \langle g\rangle$ and $H = \langle h\rangle$ and $g^m = 1 = h^n$, $(m,n) = 1$ (e.g. the binary icosahedral group), $G*H = \langle gh^{-1}\rangle$. A geometric version of this is that +6-surgery on the trefoil knot gives the same 3-manifold as the connected sum of -2-surgery and +3-surgery on trivial knots (P. Melvin).

PROBLEM 67: (Lickorish) Conjecture: Any linear subdivision on an n-simplex collapses simplicially.

Remark: True for $n \leq 3$. (Chillingworth, Proc. Cam. Phil. Soc. 63 (1967), 353-357.)

PROBLEM 68: (R. H. Bing) Does there exist a graph G such that for any imbedding $f: G \to R^3$, $f(G)$ contains a non-trivial knot?

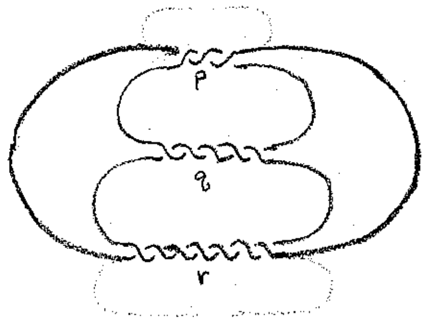
Remarks: It suffices to consider $G = C_n$ = complete graph on n-vertices. C_{12} always contains a trefoil knot if f is linear on edges (Armentrout).

Addendum

PROBLEM 4A: (Lickorish) Conjecture: Given a knot K, any band connected sum with an unknot is still a knot. This follows from the Conjecture: $\text{genus}(K) + \text{genus}(L) \leq \text{genus}(K \underset{b}{\#} L)$.

PROBLEM 18A: (A. Casson)

A) The knot



($p = -3$, $q = 5$, $r = 7$ in illustration) has Alexander polynomial 1 if p,q,r are odd and $qr+rp+pq = -1$. Is it slice?

B) The double branched covering of this knot is the Brieskorn homology sphere $(|p|,|q|,|r|)$. Does it bound a homology ball?

Remarks: An affirmative answer to A implies that (p,q,r) bounds a Z_2-homology ball.

If the Brieskorn sphere $(|2bc+1|,|2a(b-d)+1|,|2d(c-a)+1|)$ bounds a homology ball for some numbers a,b,c,d with $ad-bc = 1$, then the homology class (a,b) in some homology $S^2 \times S^2$ is representable by an embedded S^2. For example, if $(3,5,7)$ bounds, then $(2,3)$ is representable. (See PROBLEM 55.)

PROBLEM 57A: (Gluck) Let K be a knotted 2-sphere in S^4. A homotopy 4-sphere Σ^4 can be constructed by removing a tube around K and sewing it back in with a twist $T: S^1 \times S^2 \to S^1 \times S^2$ which is defined by the non-trivial element of $\pi_1(SO(3))$. Is $\Sigma^4 = S^4$?

Remark: This is equivalent to the question: is $(S^4,K) \# (CP^2,CP^1)$ pairwise diffeomorphic to (CP^2,CP^1)?, a special case of PROBLEM 57. $\Sigma^4 = S^4$ if (S^4,K) is the double of (B^4,D) where D is a ribbon disk (Melvin) or if (S^4,K) is a twist-spun knot (Gordon).